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Effects of Viscosity on Long Waves

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SUMMARY
A study of effects of viscosity on non-linear long waves is made. Beginning with the Navier–Stokes equations of motion, the long wave approximation is achieved by an expansion scheme similar to Friedrichs'. Non-linear solutions are obtained by applying the theory of relatively undistorted waves. It is found that shock formation is delayed by the viscous effect. Various conditions are obtained in determining the viscous, non-linear and radial decay effects on the solution for a shockless expansion wave-front propagating over large distances.

1. Introduction

In studying gravity waves, the fluid is generally assumed to be homogeneous and incompressible in irrotational flow without viscosity. Even with these assumptions, the basic general theory is still difficult to handle because the non-linear boundary conditions are to be applied at an unknown free surface. Special hypotheses of one kind or another which have the effect of yielding more tractable mathematical formulations have been introduced. One approximate theory results from the assumption that the wave amplitudes are small (with respect to wavelength, for example) and this theory has contributed many useful results since its first development by Airy and Stokes in the middle of the nineteenth century. Another approximation is based on the assumption that the wave-length is large compared to the vertical extent of the fluid and this is called the "long wave" (or "Shallow water") approximation of Boussinesq and Rayleigh. The long wave theory has had applications in describing tidal and other large wavelength motions. Also, since this theory contains non-linear terms in the equations it has been used to describe large amplitude waves. However, there has been some controversy about the domain of validity of the theory [8], [14], [15]. Basically there are two distinct approaches to the theory. In the first, due to Friedrichs [9], the first order result of an expansion scheme yields Airy's theory which predicts that a progressive compression wave eventually breaks; the second order result is the existence of waves of permanent form. The small parameter involved is essentially the ratio between the depth (H) and a horizontal length scale (L) (with $H/L \ll 1$). In the second, due to Ursell [15], and Lin and Clark [10], the amplitude (a) is used as a third length scale and, hence, an additional parameter (a/L). Three different domains are then classified: (1) Airy's theory belongs to the domain where $(a/L)(H/L)^{-3} \gg 1$; (2) the theory of permanent wave to $(a/L)(H/L)^{-3} = 0(1)$; and the linearized theory to $(a/L)(H/L)^{-3} \ll 1$. In all these domains $H/L \ll 1$.

It is relatively easy to introduce viscosity into the linearized approximation although the theory loses the elegance and power of potential theory and problems are more difficult to solve. A summary of elementary results of such works is given by Wehausen and Laitone [2]. The effect of viscosity on long waves has received little notice. In the book by Stoker [3], all treatments are for inviscid fluid; and for engineering problems of flood waves in rivers Chezy's empirical formula [4] is used to estimate the viscosity terms. Another approach is to assume a shear profile in the basic flow; the wave motion is, however, assumed to be governed by inviscid equations [5], [6], [7].

The serious problem encountered by considering viscosity in the long wave approximation can be seen from the nature of the governing equations. The first order inviscid equations (the shallow water equations) are hyperbolic, whilst the addition of viscous terms changes them to parabolic. However, one can argue that since the shallow water equations describe longitudinal motions of vertically-averaged profiles, the effect of viscosity in changing these profiles should be estimated. This feature of considering averaged profiles is found useful in the viscous case since it allows a formulation in which the equations remain hyperbolic.

The approach of the present study begins with the Navier–Stokes equations of motion. For the long wave approximation, we assume an expansion similar to Friedrichs'. That is to take as a small parameter the ratio of a typical depth (H_0) to a characteristic length (L_0). Assuming that the ratio of the kinematic viscosity (ν) to the product of a typical vertical velocity (V_0) and H_0 is bounded, a system of equations governing the motion of non-linear long waves is derived. The equations are integrated with respect to the vertical coordinate. The second derivative is converted into a boundary value. In this way the parabolic system is reduced to a hyperbolic system for vertically averaged values.

Two-dimensional problems are investigated. Under a small amplitude expansion, the linearized solution for a flat bottom is obtained by Fourier's method. For an uneven bottom topography whose tangent is small a simple transformation reduces the problem to the previous case. Non-linear solutions are obtained by applying the theory of relatively undistorted waves. This theory enables an estimate to be found of the effect of the non-linearity in causing shocks (bores) to form, and also estimates the effect of the non-linearity on wave propagation over large distances. The viscosity effect on shock formation is investigated. It is found that when a wave-front propagates into a quiet region of an arbitrary bottom topography the presence of viscosity is to delay the breaking of such a wave-front. In inviscid theory [11], when a shockless expansion front propagates over large distances, the non-linear terms finally dominate the solution and one result is that the solution is independent of its initial data. However, this is not necessarily so for a viscous fluid. The non-linear effect may be overshadowed by the viscous effect, and the solution decays exponentially over large distances. Conditions are obtained in determining these effects on the solution. In the inviscid limit, results derived from the present approach are in agreement with those obtained by Varley and Cumberbatch [11], Burger [23] and Stoker [3].

For three-dimensional (axially symmetric) problems, similar techniques can be employed. In addition to viscous and non-linear effect, radial decay also plays a role for a shockless expansion wave-front travelling over large distances.

The result that viscosity delays the breaking of waves approaching a shoreline is in agreemnt with experiments. The results of Mei [24] for inviscid waves give breaking too far from the shore when compared with Iverson's experiments [25].

2. Formulation of the Problem

2.1. *Basic Equations and Boundary Conditions*

We shall assume that the fluid is homogeneous, incompressible and viscous with constant coefficient of viscosity. The entire motion will be regarded as two-dimensional. In a fixed coordinate system let the positive $\bar{y}$-axis be directed upwards, i.e. opposite to the force of gravity, and let the $\bar{x}$-axis be identified with the undisturbed free surface. Let $\bar{h}(\bar{x}) \geqq 0$ be the undisturbed depth. The fluid is assumed to fill the space:

$$-\bar{h}(\bar{x}) \leqq \bar{y} \leqq \bar{\eta}(\bar{x}, \bar{t}), \qquad -\infty < \bar{x} < \infty,$$

and the vertical displacement of the free surface measured from the x-axis, $\bar{y} = \bar{\eta}(\bar{x}, \bar{t})$, is to be determined when the fluid is in motion.

Let $\bar{u}(\bar{x}, \bar{y}, \bar{t})$ and $\bar{v}(\bar{x}, \bar{y}, \bar{t})$ be the horizontal and vertical components of velocity, $\bar{p}(\bar{x}, \bar{y}, \bar{t})$ the pressure, ρ the density of the fluid, g the acceleration due to gravity, $\bar{t}$ the time and $\nu = \mu/\rho$ the kinematic coefficient of viscosity.

Let there be characteristic horizontal and vertical velocities U_0 and V_0, and a typical depth H_0. Non-dimensional quantities may then be defined as follows:

$$x=\bar{x}/L_0\,,\quad y=\bar{y}/H_0\,,\quad u=\bar{u}/U_0\,,\quad v=\bar{v}/V_0\,,$$
$$t=\bar{t}U_0/L_0\,,\quad p=\bar{p}/(\rho U_0^2)\,,\quad \eta=\bar{\eta}/H_0\,,\quad h=\bar{h}/H_0\,,$$

where L_0 is a characteristic length defined by $L_0=H_0U_0/V_0$. In addition, we introduce the parameter $\varepsilon=H_0/L_0$.

In terms of non-dimensional quantities, the Navier–Stokes equations of continuity and momentum take the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\beta\left(\frac{\partial^2 u}{\partial y^2}+\varepsilon^2\frac{\partial^2 u}{\partial x^2}\right) \tag{2}$$

$$\varepsilon^2\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right]=-\frac{\partial}{\partial y}(p+ky)+\beta\varepsilon^2\left(\frac{\partial^2 v}{\partial y^2}+\varepsilon^2\frac{\partial^2 v}{\partial x^2}\right) \tag{3}$$

where $\beta=\nu/V_0H_0$ and $k=gH_0/U_0$.

The boundary conditions are as follows:

(i) The kinematic condition on the free surface requires that

$$\frac{\partial\eta}{\partial t}+u\frac{\partial\eta}{\partial x}-v=0\ \text{ on }\ y=\eta(x,t)\,. \tag{4}$$

(ii) If we assume the surface to be free of external stress and surface tension, then the components of stress force in the fluid must vanish on the free surface, i.e.

$$\left.\begin{aligned} &p\frac{\partial\eta}{\partial x}+\beta\frac{\partial u}{\partial y}+\beta\varepsilon^2\cdot\left(\frac{\partial v}{\partial x}-2\frac{\partial u}{\partial x}\right)=0 &(5)\\ &p+\beta\varepsilon^2\left(-2\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial\eta}{\partial x}\right)+\beta\varepsilon^4\frac{\partial v}{\partial x}\frac{\partial\eta}{\partial x}=0 &(6)\end{aligned}\right\}\ \text{ on }\ y=\eta(x,t)\,.$$

(iii) On the bottom there must be no slip, i.e.

$$u=v=0\ \text{ on }\ y=-h(x) \tag{7}$$

where we assume that the bottom is rigid, impermeable and stationary.

2.2. *Long Wave Approximation*

In the following treatment, we shall assume that the depth of the fluid is small compared with a horizontal length scale, such as the wave-length (it is not necessary to assume the displacement and slope of the free surface are small). In other words, we shall assume that ε is small. We note that the feature of the long wave approximation (or shallow water theory) is that the horizontal and vertical directions are not treated in the same way (i.e. the vertical and horizontal distances are scaled differently).

There are many circumstances in nature for which such a theory provides a good model. Among such occurrences are the tides in the oceans, the large wave-length or solitrary waves, waves approaching beaches including the breaking of such waves [3], [16], [17], [18], [19], and airflow over mountains [1].

Our present approach is somewhat closer to Friedrichs' [9]. We adopt an iterative scheme for equations (1)–(7) which is based on $\varepsilon\ll 1$ and shall consider only the first term. In addition to ε, in later sections we consider a typical amplitude δ as another parameter. Also, the inclusion of viscosity brings added complexity because of the new parameter $\beta=\nu/V_0H_0$.

We shall assume that β is not too large, e.g. $\beta=0(1)$, such that $\varepsilon^2\beta$ terms in equations (1)–(7)

are negligible. Let us take a numerical example for the problem of a flood in a model of the Ohio River (p. 488 of [3]). For a constant slope of 0.5 ft/min, a constant breath of 1000 ft and an initial velocity of the water 2.38 mph, the propagation speed of small disturbance corresponding to the depth of 20 ft is 17.3 mph. If we take $H_0 = 20$ ft, $\nu = 1.08 \cdot 10^{-5}$ ft^2/sec (at the temperature 20°C) and V_0 to be $(0.5/5280) \cdot 2.38$ (this choice of V_0 gives the largest value of β) then $\beta = 4.14$. Hence, it seems to be reasonable to assume that β is bounded. In what follows we shall ignore the ε^2 and $\varepsilon^2\beta$ terms.

3. Method of Integral Relations

In the limit $\varepsilon \to 0$ the second derivative of u with respect to x is omitted in equation (2). Equations (1) and (2) are now integrated with respect to y. In (2) the second derivative of u with respect to y is converted into a boundary value. In this way the parabolic system is reduced to a hyperbolic system for y-averaged values of u and u^2. Various progressing wave solutions may then be investigated.

By integrating (3) and using (6) the pressure p is given as in hydrostatics by

$$p = k(\eta - y)\,. \tag{8}$$

This relation can be taken as a starting point for a derivation of inviscid shallow water theory.

In the limit $\varepsilon \to 0$ equation (2) may be rewritten as

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) = -k\,\frac{\partial \eta}{\partial x} + \beta\,\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \tag{9}$$

where (1) and (8) have been used.

Integrating (9) with respect to y from $-h(x, t)$ to $\eta(x, t)$ and using relations such as

$$\frac{\partial}{\partial x}\int_{-h}^{\eta} u\,dy = u\Big|_{y=\eta} \cdot \frac{\partial \eta}{\partial x} + u\Big|_{y=-h} \cdot \frac{\partial h}{\partial x} + \int_{-h}^{\eta} \frac{\partial u}{\partial x}\,dy$$

and boundary conditions (4)–(7) we obtain

$$\frac{\partial}{\partial t}\int_{-h}^{\eta} u\,dy + \frac{\partial}{\partial x}\int_{-h}^{\eta} u^2\,dy = -k(\eta + h)\frac{\partial \eta}{\partial x} - \beta\,\frac{\partial u}{\partial y}\Big|_{y=-h} \tag{10}$$

Similarly, integration of (1) with respect to y and application of (4)–(7) yield

$$\frac{\partial}{\partial x}\int_{-h}^{\eta} u\,dy + \frac{\partial \eta}{\partial t} = 0\,. \tag{11}$$

If we consider in (10) and (11) a series

$$u = \sum_{m=1}^{j} a_m(x, t) f_m(y)$$

where the f_m functions are chosen in some appropriate way, there results a system of $2j$ simultaneous partial differential equations for the y-averaged value of u, i.e., $a_m(x, t)$. This approach is called the method of integral relations and is used in hypersonic theory to calculate numerically the solution of flow between a shock and a blunt body [12] (the shock corresponds to the unknown free surface). Here we are interested in the analytical properties of the solution and we consider only one term of such a series. In this way we attempt to get information on the viscous decay of waves by approximating in a crude way the vertical profile of u. The important thing is that the viscous bondary condition at the bottom is being satisfied.

We consider a velocity profile

$$u = \frac{N+1}{N} \cdot \frac{U}{\eta + h}\left[1 - \frac{(\eta - y)^N}{(\eta + h)^N}\right] \tag{12}$$

where $N = 1, 2, 3, \ldots$ and U is an unknown function of x and t. Equations (10) and (11) now become

$$\frac{\partial U}{\partial t} + \frac{4(N+1)}{2N+1}\frac{U}{\eta+h}\frac{\partial U}{\partial x} - \frac{2(N+1)}{2N+1}\frac{U^2}{(\eta+h)^2}\frac{\partial \eta}{\partial x} =$$

$$= \frac{2(N+1)}{2N+1}\frac{U^2}{(\eta+h)^2}\frac{dh}{dx} - k(\eta+h)\frac{\partial \eta}{\partial x} - \beta(N+1)\frac{U}{(\eta+h)^2}, \quad (13)$$

$$\frac{\partial U}{\partial x} + \frac{\partial \eta}{\partial t} = 0 \,. \quad (14)$$

The above equations are quasi-linear and hyperbolic. The number of independent variables is reduced to two (x and t) and the dependent variables to two (U and η).

Before we conclude this section, an important observation must be made. Let us consider the relation between the present approach and the inviscid solution. The basic assumption for inviscid shallow water theory is that the y-component of the acceleration of the fluid particles has a negligible effect on the pressure p, or what amounts to the same thing, that $\partial p/\partial x$ is independent of y [3]. It follows that the x-component of the acceleration of the fluid particles is also independent of y; and hence u, the x-component of velocity is also independent of y for all t if it was, at any time, say at $t=0$. To meet this requirement the velocity profile of u should be square for fixed x and t. An inclusion of this as a special case of our general profile (12) can be achieved by letting N approach infinity in (12) after we exclude the point of discontinuity ($y=-h$) which gives $u=0$. Physically, $u=0$ at $y=-h$ corresponds to $\beta\neq 0$. For the inviscid case ($\beta=0$), this no-slip boundary condition must be relaxed. Hence, after setting $\beta=0$, we have

$$u = \frac{U}{\eta+h} \text{ as } N\to\infty$$

which is independent of y as desired. Therefore, the inviscid limit may be obtained by letting $N\to\infty$ (after having set $\beta=0$).

4. Small Amplitude Expansion

4.1. *Flat Bottom*

Let δ be a typical non-dimensional amplitude and assume it is small. This enables us to introduce the following expansions in powers of δ. For simplicity we also assume $h=1$. Then we can write

$$\left.\begin{aligned} U(x,t) &= \sum_{i=1}^{\infty} U^{(i)}(x,t)\delta^i \\ \eta\,(x,t) &= \sum_{i=1}^{\infty} \eta^{(i)}\,(x,t)\delta^i \end{aligned}\right\} \quad (15)$$

Substituting (15) into equations (13) and (14), we obtain, by equating coefficients of like powers of δ, equations for the successive coefficients in the series. The terms of first order yield the equations

$$\frac{\partial U^{(1)}}{\partial t} + k\frac{\partial \eta^{(1)}}{\partial x} = -(N+1)\beta U^{(1)} \quad (16)$$

$$\frac{\partial U^{(1)}}{\partial x} + \frac{\partial \eta^{(1)}}{\partial t} = 0 \quad (17)$$

which are equivalent to

$$\frac{\partial^2 \eta^{(1)}}{\partial t^2} - k\frac{\partial^2 \eta^{(1)}}{\partial x} = -(N+1)\beta\frac{\partial \eta^{(1)}}{\partial t}. \quad (18)$$

For initial conditions: $\eta^{(1)}(x,0)=F(x)$, $\eta_t^{(1)}(x,0)=G(x)$, the solution of (18) is

$$\eta^{(1)}=\tfrac{1}{2}\,e^{-\frac{1}{2}(N+1)\beta t}\Big\{F(x+\sqrt{k}t)+F(x-\sqrt{k}t)+$$
$$+\frac{1}{\sqrt{k}}\int_{x-\sqrt{k}t}^{x+\sqrt{k}t}F(\alpha)\cdot\frac{\partial}{\partial t}I\left(\frac{(N+1)\beta}{2\sqrt{k}}\sqrt{kt^2-(x-\alpha)^2}\right)d\alpha+$$
$$+\frac{(N+1)\beta}{2\sqrt{k}}\int_{x-\sqrt{k}t}^{x+\sqrt{k}t}\left[G(\alpha)+\frac{(N+1)\beta}{2}F(\alpha)\right]\cdot I\left(\frac{(N+1)\beta}{2\sqrt{k}}\sqrt{kt^2-(x-\alpha)^2}\right)d\alpha\Big\} \quad (19)$$

where $I(x)$ is the modified Bessel function of the first kind

$$I(x)=J_0(ix)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-x\cos\omega}d\omega=\sum_{s=0}^{\infty}\frac{1}{(s!)^2}\left(\frac{x}{2}\right)^{2s}.$$

The first two terms in (19) show that the waves propagate with velocity $\sqrt{k}$ which is the same as if $\beta=0$ (non-viscous). The waves are exponentially damped, and the damping is faster as N increases. In the limit of $\beta\to 0$ we have inviscid waves. The integral terms indicate the effect coming from all points where F and G are not both zero and within distance $\sqrt{kt}$.

4.2. *Uneven Bottom*

In the case that $h(x)$ is not constant, equations (13) and (14) are linearized to yield

$$\left.\begin{aligned}&\frac{\partial U}{\partial t}+kh\frac{\partial\eta}{\partial x}=-\frac{\beta(N+1)}{h^2}U\\&\frac{\partial U}{\partial x}+\frac{\partial\eta}{\partial t}=0\,.\end{aligned}\right\} \quad (20)$$

If the magnitude of $h'(x)$ is of a relatively small order, the first order solutions of (13) and (14) can be obtained by a transformation as indicated below.
Let

$$T=\frac{(N+1)\beta}{h^2}t\,,\qquad X=\frac{(N+1)\beta}{\sqrt{k}}\int_0^x h^{-\frac{5}{2}}dx$$

and

$$\bar{\eta}(X,T)=\eta(x,t)\,.$$

If h' is negligible, equation (20) may be written as

$$\bar{\eta}_{TT}-\bar{\eta}_{XX}=-\bar{\eta}_T \quad (21)$$

which is of the same form as equation (18) and can similarly be solved. We illustrate the effect of an uneven bottom on the wave propagation speed and the attenuation factor by considering an example.

Example. For $h=(1\pm\sigma x)$ where $0<\sigma x<1$ the wave propagation speed is $\bar{S}_\pm=\sqrt{k(1\pm\sigma x)}$ (for $0<\sigma x<1$) and the attenuation factor is $\bar{A}_\pm=\exp\{-(N+1)\beta t/2(1\pm\sigma)^2\}$. By denoting the propagation speed and attenuation factor for $h=1$ by $\bar{S}_0=\sqrt{k}$ and $\bar{A}_0=\exp[-\frac{1}{2}(N+1)\beta t]$, we obtain the following inequalities: $\bar{S}_-<\bar{S}_0<\bar{S}_+$, $\bar{A}_->\bar{A}_0>\bar{A}_0>\bar{A}_+$. The wave propagation speed increases as the depth increases and the damping is more severe.

4.3. *Discussion of Solution*

The above perturbation is based on the linearized solution which can be expected to be a good approximation for small amplitude waves except in certain regions. Linearized solutions are not good for large distances; non-linear effects provide a cumulative influence which ultimately invalidates the linearized solution. The solution is not adequate in regions where shock waves (or bores) are forming. Under the above procedure we are unable to control the region of validity of the solution. In the following section we shall obtain a non-linear solution by applying the theory of relatively undistorted waves developed by Varley and Cumberbatch.

5. Slowly-Varying Solution

The theory of relatively undistorted waves provides solutions describing phenomena where the terms "slowly-varying" or "high-frequency" may be applied to the wave motion (see Cumberbatch [22], and Varley and Cumberbatch [13]. The technique is based on a scheme of successive approximations to a system of hyperbolic equations. The theory, which makes no assumptions on the magnitude of a disturbance, is exact for simple waves, acceleration fronts and for the formation of shocks.

Let $(f_1, f_2) \equiv (\eta, U)$. A wave is said to be relatively undistorted in f_1 and f_2 with respect to the variable x if there exists a family of propagating surfaces $\alpha(x, t)=$constant, called wavelets, such that the magnitude of the rate of change of f_1 and f_2 with x moving the wavelets is small compared with the magnitude of the rate of change of f_1 and f_2 with x at fixed t. If $t=\hat{t}(x, \alpha)$ denotes the time of arrival of the wavelet "α" at x and if we write

$$f_i(x, t) = f_i(x, \hat{t}(x, \alpha)) = \hat{f}_i(x, \alpha)\,, \qquad i = 1, 2$$

then, in a relatively undistorted wave,

$$\left|\frac{\partial \hat{f}_i}{\partial x}\right| \ll \left|\frac{\partial f_i}{\partial x}\right| \qquad i = 1, 2\,.$$

It can be shown that the curves $\alpha=$constant are necessarily characteristic curves of the equations governing f_i.

5.1. *Large ω Expansion*

Equations (13) and (14) can be rewritten as

$$\frac{\partial \hat{U}}{\partial \alpha} + \left[\frac{4(N+1)}{2N+1}\frac{\hat{U}}{\xi}\frac{\partial \hat{U}}{\partial \alpha} + \frac{\partial \hat{\eta}}{\partial \alpha}\left(k\xi - \frac{2(N+1)}{2N+1}\frac{\hat{U}^2}{\xi^2}\right)\right]\frac{\partial \alpha}{\partial x}\Big/\frac{\partial \alpha}{\partial t} =$$

$$= \left[\frac{2(N+1)}{2N+1}\frac{\hat{U}^2}{\xi^2}\frac{dh}{dx} - (N+1)\beta\,\frac{\hat{U}}{\xi^2} - \left(k\xi - \frac{2(N+1)}{2N+1}\frac{\hat{U}^2}{\xi^2}\right)\cdot\frac{\partial \hat{\eta}}{\partial x} - \frac{4(N+1)}{2N+1}\frac{\hat{U}}{\xi}\frac{\partial \hat{U}}{\partial x}\right]\Big/\frac{\partial \alpha}{\partial t} \tag{22}$$

$$\frac{\partial \hat{\eta}}{\partial \alpha} + \frac{\partial \hat{U}}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial x}\Big/\frac{\partial \alpha}{\partial t} = -\frac{\partial \hat{U}}{\partial x}\Big/\frac{\partial \alpha}{\partial t} \tag{23}$$

where $\xi(x, \alpha) = h(x) + \hat{\eta}(x, \alpha)$, and in what follows we shall drop the "hat" for the sake of convenience.

An expression based on taking the right-hand sides of (22) and (23) as small is now achieved by the formal asymptotic expansions:

$$U = U_0(x, \alpha) + \frac{1}{\omega}U_1(x, \alpha) + \ldots \tag{24}$$

$$\eta = \eta_0(x, \alpha) + \frac{1}{\omega}\eta_1(x, \alpha) + \ldots \tag{25}$$

where

$$t = \frac{\alpha}{\omega} + \int_0^x \left[\frac{2(N+1)}{2N+1} \frac{U}{\xi} + \sqrt{\frac{2(N+1)}{(2N+1)^2} \frac{U^2}{\xi^2} + k\xi} \right]^{-1} dx \tag{26}$$

defines the α = constant curves as the forward characteristics, and ω is a large non-dimensional frequency parameter. This asymptotic expansion involving the large parameter ω describes situations where there are a large number of waves in an attenuation length and any single wave changes slowly as it travels, i.e., it is relatively undistorted. It also provides the solution for the simple wave (here h = constant, $\beta = 0$ are necessary for this) and the solution near a wave front. As we shall see in later sections that in the limit $\omega \to \infty$ shocks will form arbitrarily close to the input station. The condition

$$\omega A_1 = 0(1) \text{ as } \omega \to \infty \tag{27}$$

(where A_1 is the amplitude of the pulse at the input station) must be imposed to get a finite shockless domain of validity for the relatively undistorted solution. The above restriction on A_1 implies that we are dealing with small amplitude but finite acceleration waves.

Since only the first terms of the expansions (24) and (25) will be calculated, the zero subscript will be omitted. The equations governing the first terms are as follows, from (23) and (26)

$$\frac{\partial U}{\partial \alpha} = \left[\frac{2(N+1)}{2N+1} \frac{U}{\xi} + \sqrt{\frac{2(N+1)}{(2N+1)^2} \frac{U^2}{\xi^2} + k\xi} \right] \frac{\partial \eta}{\partial \alpha} \tag{28}$$

and from the compatibility relation between (22) (23) (26)

$$-\frac{2(N+1)}{2N+1} \frac{U^2}{\xi^2} \frac{dh}{dx} + (N+1)\beta \frac{U}{\xi^2} + \left(k\xi - \frac{2(N+1)}{2N+1} \frac{U^2}{\xi^2} \right) \frac{\partial \eta}{\partial x} =$$

$$= \left(\frac{2(N+1)}{2N+1} \frac{U}{\xi} - \sqrt{\frac{2(N+1)}{(2N+1)^2} \frac{U^2}{\xi^2} + k\xi} \right) \frac{\partial U}{\partial x}. \tag{29}$$

Since h is not a function of α and $\xi = h + \eta$, by (28) we have

$$\left(\frac{dU}{d\xi} \right)_{x=\text{constant}} = \frac{2(N+1)}{2N+1} \frac{U}{\xi} + \sqrt{\frac{2(N+1)}{(2N+1)^2} \frac{U^2}{\xi^2} + k\xi}\,. \tag{30}$$

The solution for equation (30) in implicit form is

$$A(x)\xi^{a_1} = \frac{(\sqrt{U^2 + b_2\xi^3} + U)^{a_2} \cdot (\sqrt{U^2 + b_2\xi^3} - U)^{a_3}}{(\sqrt{U_2 + b_2\xi^3})^{a_4} (U + b_1\sqrt{U^2 + b_2\xi^3})^{a_5}} \tag{31}$$

where $A(x)$ is a function of x to be determined by a boundary condition and where

$$b_1 = \frac{2\sqrt{2(N+1)}}{-2N+1}, \qquad b_2 = \frac{k(2N+1)^2}{2(N+1)},$$

$$a_1 = \frac{-4N^2 + 8N + 3}{2(2N+1)}, \qquad a_2 = \sqrt{2(N+1)} - N + \tfrac{1}{2},$$

$$a_3 = \sqrt{2(N+1)} + 5N + \tfrac{7}{2}, \quad a_4 = 2\sqrt{2(N+1)} + 2N - 1, \quad a_5 = -2N + 1.$$

In principle, we can solve for η in equation (29) by replacing U by a function $U(\xi)$ through relation (31). However, the dependence of U on ξ in (31) is so involved and its use in producing further results is forbidden by complexities. For simplicity, we shall consider some special cases.

5.2. *First Order Solution*

A useful approximate solution is obtained by taking the linearized solution of (29) and determining the characteristics to the first order.

Let δ be a non-dimensional typical amplitude taken as small. Consider the case $U=0(\delta)$ and $\eta=0(\delta)$.

Then from equation (31) we can express U in terms of η as

$$U=\left[\frac{2(N+1)}{2N+1}A(x)h^{(2N+1)^{-1}}+\frac{3(2N+1)}{2N-1}\sqrt{k}h^{\frac{1}{2}}\right]\eta+ \\ +\left[A(x)h^{2(N+1)/(2N+1)}+\frac{2(2N+1)}{2N-1}\sqrt{k}h^{\frac{3}{2}}\right]+0(\delta^2)\,. \qquad (32)$$

If we consider waves propagating into a quiet region, then ahead of the wave $U\equiv 0$, $\eta\equiv 0$ on $\alpha=0$.

This implies that

$$A(x)=-\frac{2(2N+1)}{2N-1}\sqrt{k}h^{(2N-1)/2(2N+1)} \qquad (33)$$

and from (32)

$$U=\sqrt{kh}\,\eta\,. \qquad (34)$$

Now equation (29) is linearized to

$$\frac{(N+1)\beta U}{h^2}+kh\frac{\partial\eta}{\partial x}+\sqrt{kh}\frac{\partial U}{\partial x}+0(\delta^2)=0\,. \qquad (35)$$

Substituting (34) into (35) yields

$$\frac{\partial\eta}{\partial x}+\left(\frac{(N+1)\beta}{2\sqrt{k}h^{\frac{5}{2}}}+\frac{1}{4}\frac{h'}{h}\right)\eta=0 \qquad (36)$$

which gives

$$\eta=B(\alpha)\exp\left\{-\frac{1}{2}\int_0^x\left[\frac{(N+1)\beta}{\sqrt{k}h^{\frac{5}{2}}}+\frac{1}{2}\frac{h'}{h}\right]dx\right\} \qquad (37)$$

where $B(\alpha)$ is to be determined by the boundary conditions. For example, $\eta=A_1\sin\alpha$ on $x=0$ gives $B(\alpha)=A_1\sin\alpha$ ($A_1=$constant). Now, by (26), (34) and (37) we obtain

$$t=\frac{\alpha}{\omega}+\int_0^x\frac{dx}{\sqrt{kh}+\frac{6N+5}{2(2N+1)}\sqrt{\frac{k}{h}}B(\alpha)\exp\left\{-\frac{1}{2}\int_0^x\left[\frac{(N+1)\beta}{\sqrt{k}h^{\frac{5}{2}}}+\frac{1}{2}\frac{h'}{h}\right]dx\right\}}\,. \qquad (38)$$

In the linear approximation in secs. 4.1 and 4.2 the characteristics are straight lines. However, (38) shows that the characteristics of the first order slowly-varying solution are, in general, not straight.

5.3. *Formation of Shock Waves*

The shock positions (or critical distances) are obtained by solutions of $t_\alpha=0$. Differentiating (26) with respect to α, the shock position, x_s, is given by:

$$0 = \frac{1}{\omega} - \int_0^{x_s} \frac{\frac{\partial}{\partial \alpha}\left[\frac{2(N+1)}{2N+1}\frac{U}{\xi} + \sqrt{\frac{2(N+1)}{(2N+1)^2}\frac{U^2}{\xi^2} + k\xi}\right]}{\left[\frac{2(N+1)}{2N+1}\frac{U}{\xi} + \sqrt{\frac{2(N+1)}{(2N+1)^2}\frac{U^2}{\xi^2} + k\xi}\right]^2} dx \tag{39}$$

where U and ξ are solutions of (28) and (29) subjected to appropriate boundary conditions. The corresponding critical time is determined by equation (26) by substituting x_s for x.

In the remaining discussion, we shall use the first order approximations in locating shock formation which will give estimates correct to the first order in the amplitude. Now (39) becomes

$$0 = \frac{1}{\omega} - \int_0^{x_s} \frac{B'(\alpha)\left[\frac{6N+5}{2(2N+1)}\sqrt{\frac{k}{h}}\exp\{F(x)\}\right]dx}{\left[\sqrt{kh} + \frac{6N+5}{2(2N+1)}\sqrt{\frac{k}{h}}B(\alpha)\exp\{F(x)\}\right]^2} \tag{40}$$

where

$$F(x) = -\tfrac{1}{2}\int_0^x \left[\frac{(N+1)\beta}{\sqrt{k}h^{\frac{5}{2}}} + \tfrac{1}{2}\frac{h'}{h}\right]dx\,. \tag{41}$$

We shall consider the propagation of a wave-front of an arbitrary initial form into a quiet region in which the depth varies in an arbitrary way. For the boundary condition $B(\alpha)=0$ and $B'(\alpha)=A_1>0$ (for example, $B(\alpha)=A_1 \sin \omega t$ on $x=0$ at $t=0$) the shock position, x_s, is

$$0 = \frac{1}{\omega} - \frac{(6N+5)A_1}{2(2N+1)\sqrt{k}}\int_0^{x_s} h^{-\frac{3}{2}}\exp\{F(x)\}\,dx\,. \tag{42}$$

If we set $\beta=0$ and then let $N\to\infty$, we obtain the shock position, $x_s^{(0)}$, for inviscid fluid from

$$0 = \frac{1}{\omega} - \frac{3A_1 h_1^{\frac{1}{4}}}{2\sqrt{k}}\int_0^{x_s^{(0)}} h^{-7/4}\,dx \tag{43}$$

where $h_1=h(0)$. Equation (43) agrees with the result given by Varley and Cumberbatch in their investigation of non-linear theory of wave-front propagation [11], and also recently by Burger [23].

To investigate the effect of viscosity on shock formation, we shall consider some special cases.

Case 1. Flat Bottom

When $h=\text{constant}=h_1$, (42) gives

$$x_s = -\frac{2\sqrt{k}h_1^{\frac{5}{2}}}{(N+1)\beta}\log\left[1 - \frac{(2N+1)(N+1)\beta}{(6N+5)h_1\omega A_1}\right]. \tag{44}$$

The inviscid limit (i.e. $\beta=0$) is

$$x_s^{(0)} = \frac{2(2N+1)\sqrt{k}h_1^{\frac{3}{2}}}{(6N+5)A_1\omega}\,. \tag{45}$$

If we denote $C_0=\sqrt{kh_1}$ as the initial speed of propagation of the wave-front then as $N\to\infty$ we obtain

$$x_s^{(0)} = \frac{2C_0^3}{3kA_1\omega} \tag{46}$$

and the corresponding critical time

$$t_x^{(0)} = \frac{2C_0^2}{3kA_1\omega}$$

which coincide with the results obtained by Stoker [20].

For large ω and small β the value of the second term inside the square brackets on the right-hand side of (44) is between 0 and 1 (N is usually taken to be 2). Since $\log(1-x) < -x$ for $|x| < 1$ $(x \neq 0)$, it can easily be seen from (44) and (45) that

$$x_s^{(0)} < x_s$$

provided that x_s exists. This means that the presence of the viscosity is to delay the formation of shock at the wave-front.

We note that the linearized theory limit is $A_1 \to 0$, this yields

$$x_s^{(0)} = \infty \text{ and } x_s = \infty$$

i.e. infinitesimal waves never break.

We also note that shock can be prevented by viscosity, i.e.

$$x_s = \infty \text{ if } \beta = \frac{(6N+5)h_1 A_1 \omega}{(2N+1)(N+1)}.$$

Case 2. Beach Problem

We consider a non-uniformly sloping beach. Let us assume that at time $t=0$ a wave-front exists in region $x \leqq 0$ and the water is undisturbed in region $0 < x \leqq d$, where $x=$d and $h(d)=0$ is the shoreline.

Suppose that the shape of the bottom near $x=d$ can be represented by a single algebraic form $h=C_1(d-x)^q$ with $q>0$ and $C_1>0$. Then the shock position is given by

$$0 = \frac{1}{\omega} - \frac{(6N+5)A_1}{2(2N+1)\sqrt{k}} \int_0^{x_s} C_1^{-7/4}(d-x)^{-(7/4)q} \exp\left\{-\frac{(N+1)\beta}{2\sqrt{k}\,C_1^{\frac{5}{2}}} \int_0^x \frac{dx}{(d-x)^{(\frac{5}{2})q}}\right\} dx\,. \quad (47)$$

We can see from (47) that the wave may break at some $x < d$ if $q \geqq \frac{4}{7}$. If $\frac{2}{5} < q < \frac{4}{7}$ the wave will not break in $x < d$ provided that $\beta \neq 0$. If $q < \frac{2}{5}$, whether or not the wave breaks at $x < d$ depends on the behaviour of $h(x)$ for $0 \leqq x < d$. However, all boreless compression waves break at the shoreline (this can be seen in section 5.4 or in [11]).

If $x_s = x_s^{(0)}$ when $\beta=0$, it is obvious from (47) that

$$x_s^{(0)} < x_s$$

provided that x_s exists. This implies that if the wave breaks when climbing up a sloping beach the viscosity effect is to postpone the breaking.

For the special case when $\beta=0$ and $q=1$ (i.e. a uniformly sloping beach $h=h_1(d-x)/d$), as $N \to \infty$

$$x_s^{(0)} = d - \frac{3\omega A_1 d^{7/4}}{\sqrt{k}\,h_1^{\frac{3}{2}} + 3A_1 \omega d}. \quad (48)$$

By contrast, the linearized theory limit is

$$x_s^{(0)} = d$$

i.e. waves only break at the shoreline.

5.4. *Wave Propagation Over Large Distances*

The non-linear effect investigated in the previous section are concerned with the steepening of wavefronts and it was seen clearly that the effect of viscosity is to weaken this steepening. Another finite-amplitude effect is a cumulative influence in propagation over large distances. The non-linear terms finally dominate the solution and one result is that (to the first order) the solution is independent of its initial data. What is of interest now is the weakening effect of viscosity on this property.

At a wave-front, propagating in a direction of increasing x with boundary condition $B(\alpha)=0$ and $B'(\alpha)=A_1$ on $\alpha=0$, the slope of the free surface, γ, is obtained from (26) and (37)

$$\gamma = -\left(\frac{\partial \eta}{\partial x}\right)_{t=\text{constant}} \tag{49}$$

or

$$\gamma = \frac{A_1 h_1^{\frac{1}{4}}}{\sqrt{k}\, h^{\frac{3}{4}}} \exp\{G(x)\} \left[\frac{1}{\omega} - \frac{(6N+5)A_1 h_1^{\frac{1}{4}}}{2(2N+1)\sqrt{k}} \int_0^x h^{-7/4} \exp\{G(x)\} dx\right]^{-1} \tag{50}$$

where

$$G(x) = -\frac{(N+1)\beta}{2\sqrt{k}} \int_0^x h^{-\frac{5}{2}} dx \tag{51}$$

and $h_1 = h(0)$.

A shock begins to form (i.e. $|\gamma| \to \infty$) when the term in square brackets in (50) has a zero. In linear theory this term is replaced by $1/\omega$.

In the inviscid limit (i.e. $\beta=0$ and then $N\to\infty$), (50) gives

$$\gamma^{(0)} = \frac{A_1 h_1^{\frac{1}{4}}}{\sqrt{k}\, h^{\frac{3}{4}}} \left[\frac{1}{\omega} - \frac{3A_1 h_1^{\frac{1}{4}}}{2\sqrt{k}} \int_0^x h^{-7/4} dx\right]^{-1} \tag{52}$$

which coincides with the result obtained by Varley and Cumberbatch [11].

If we consider an expansion wave (i.e. $A_1 < 0$) propagating into a region of unlimited extent and bounded depth, as pointed out by Varley and Cumberbatch, (52) shows that a shock never forms, $\gamma^{(0)}$ decays, and the non-linear terms finally dominate. For large value of x we have

$$\gamma^{(0)} \sim -\tfrac{2}{3} h^{-\frac{3}{4}} \left[\int_0^x h^{-7/4} dx\right]^{-1} \tag{53}$$

which is independent of A_1. Hence, asymptotically an expansion front "forgets" detail of the conditions at any finite time. This aspect of non-linearity is in direct contradiction with linear theory.

The effect of viscosity on this property can be seen from (50); the integral in the square brackets is no longer the dominating term for large values of x. The relative magnitudes between the various parameters in (50) dictate the answer. Let us denote (assume it exists)

$$C_n = \frac{(6N+5)A_1 h_1^{\frac{1}{4}}}{2(2N+1)\sqrt{k}} \int_0^x h^{-7/4} \exp\{G(x)\} dx\,.$$

Thus if $1/\omega \ll C_n$, (50) gives

$$\gamma \sim -\frac{2(2N+1)}{(6N+5)} \exp\{G(x)\} \left[\int_0^x h^{-7/4} \exp\{G(x)\}\, dx\right]^{-1}$$

which is independent of A_1. In this case initial data has no effect on the solution over a large distance. A combination of viscous and non-linear effects dominate the solution. If $1/\omega \gg C_n$ then we have

$$\gamma \sim \frac{A_1 h_1^{\frac{1}{4}} \omega}{\sqrt{k}\, h^{\frac{3}{4}}} \exp\left\{-\frac{(N+1)\beta}{2\sqrt{k}} \int_0^x h^{-\frac{5}{2}} dx\right\}$$

and the motion is viscous dominated.

We note that the viscous damping effect disappears when

$$h = \left[\frac{q(N+1)\beta}{\sqrt{k}\, h'}\right]^{\frac{2}{3}}$$

i.e.

$$h = \left(\frac{5q(N+1)\beta}{2\sqrt{k}} x + h_1^{\frac{5}{2}}\right)^{2/5}$$

where $q \neq 0$ is a real number such that the integral in (50) exists. In this case (50) becomes

$$\gamma = \frac{A_1 h_1^a}{\sqrt{k} h^b} \left[\frac{1}{\omega} - \frac{(6N+5) A_1 h_1^a}{2(2N+1)\sqrt{k}} \int_0^x h^{-c} dx\right]^{-1} \tag{55}$$

where

$$a = \tfrac{1}{4} + \frac{1}{2q}, \quad b = \tfrac{3}{4} + \frac{1}{2q}, \quad c = \tfrac{7}{4} + \frac{1}{2q}.$$

6. *Axially Symmetric Free Surface Flow*

Let us consider an axially symmetric free surface flow; that is a flow in which the distribution of all quantities is completely described by its pattern in a meridian plane. Details of this derivation can be found in Wen [26]. Employing similar techniques as in the previous sections, the first order slowly-varying solution gives the slope, S, of the free surface at a wave-front as it propagates into a quiet region.

$$S = \frac{A_1 h_1^{\frac{3}{4}} c^{\frac{1}{2}}}{\sqrt{k} h^{\frac{3}{4}} r^{\frac{1}{2}}} \exp\{G(r)\} \left[\frac{1}{\omega} - \frac{(6N+5) A_1 h_1^{\frac{3}{4}} c^{\frac{1}{2}}}{2(2N+1)\sqrt{k}} \int_c^r h^{-7/4} r^{-\frac{1}{2}} \exp\{G(r)\} dr\right]^{-1} \tag{56}$$

where

$$G(r) = -\frac{(N+1)\beta}{2\sqrt{k}} \int_c^r h^{-\frac{5}{2}} dr, \quad c = \text{constant} > 0, \text{and } h_1 = h(c).$$

The shock position r_s is given by

$$0 = \frac{1}{\omega} - \frac{(6N+5) A_1 h_1^{\frac{3}{4}} c^{\frac{1}{2}}}{2(2N+1)\sqrt{k}} \int_c^{r_s^{(0)}} r^{-\frac{1}{2}} h^{-7/4} \exp\left\{-\frac{(N+1)\beta}{2\sqrt{k}} \int_c^r h^{-\frac{5}{2}} dr\right\} dr.$$

In the inviscid limit (i.e. $\beta = 0$ and then $N \to \infty$) the shock position, $r_s^{(0)}$, is given by

$$0 = \frac{1}{\omega} - \frac{3 A_1 h_1^{\frac{3}{4}} c^{\frac{1}{2}}}{2\sqrt{k}} \int_c^{r_s} r^{-\frac{1}{2}} h^{-7/4} dr.$$

Comparing r_s with $r_s^{(0)}$ for a compression wave-front it is obvious that $r_s^{(0)} < r_s$ provided r_s exists. This is to say that the viscous effect is to delay the shock formation.

For a special case of a flat bottom (i.e. $h = h_1$), the shock position is found to be

$$r_s^{(0)} = \frac{(3 A_1 \omega c + \sqrt{k} h_1^{\frac{3}{2}})^2}{9 A_1^2 \omega^2 c}$$

and the corresponding critical time

$$t_s^{(0)} = \frac{1}{\sqrt{kh}} \left[\frac{(3 A_1 \omega c + \sqrt{k} h_1^{\frac{3}{2}})^2}{9 A_1^2 \omega c} - c\right].$$

In the inviscid limit of (56) we have

$$S^{(0)} = \frac{A_1 h_1^{\frac{3}{4}} c^{\frac{1}{2}}}{\sqrt{k} h^{\frac{3}{4}} r^{\frac{1}{2}}} \left[\frac{1}{\omega} - \frac{3 A_1 h_1^{\frac{3}{4}} c^{\frac{1}{2}}}{2\sqrt{k}} \int_c^r h^{-7/4} r^{-\frac{1}{2}} dr\right]^{-1} \tag{57}$$

which coincides with Varley and Cumberbatch's result [11]. At a shockless expansion wave-front propagating into an unlimited extent and bounded depth the non-linear term in (57) will eventually dominate for large values of r, i.e.

$$S^{(0)} \sim -\tfrac{2}{3} h^{-\frac{3}{4}} r^{-\frac{1}{2}} \left[\int_c^r h^{-7/4} r^{-\frac{1}{2}} \, dr \right]^{-1}$$

which is independent of A_1 and has radial decay. For a flat bottom (i.e. $h = h_1$)

$$S^{(0)} \sim -\tfrac{1}{3} h_1^{-\frac{5}{2}} r^{-1}$$

which shows that the radial decay and non-linear effect dictate the solution.

When viscosity is included the situation is more complex. Let Cn be the integral term in the square brackets of (56). If $0(1/\omega) \ll 0(Cn)$ for large r, we get

$$S \sim -\frac{2(2N+1)}{(6N+5)} h^{-\frac{3}{4}} r^{-\frac{1}{2}} \exp\{G(r)\} \left[\int_c^r h^{-7/4} r^{-\frac{1}{2}} \exp\{G(r)\} \, dr \right]^{-1}$$

in which the non-linearity, viscosity and radial decay all play roles. If $0(1/\omega) \gg 0(Cn)$, (56) gives

$$S \sim \frac{A_1 \omega h_1^{\frac{1}{4}} c^{\frac{1}{2}}}{\sqrt{k} h^{\frac{3}{4}} r^{\frac{1}{2}}} \exp\{G(r)\}$$

in which the effect of viscosity is dominant but radial decay also plays a role.

We note that the effects of viscosity and radial decay cancel when h satisfies:

$$\frac{1}{q} \frac{h'}{h} + \frac{1}{r} + \frac{(N+1)\beta}{\sqrt{k} h^{\frac{5}{2}}} = 0$$

where $q \neq 0$ is a real number. Then (56) becomes

$$S = \frac{A_1 h_1^a}{\sqrt{k} h^b} \left[\frac{1}{\omega} - \frac{3 A_1 h_1^a}{2\sqrt{k}} \int_c^r h^{-l} \, dr \right]^{-1}$$

where $a = \frac{1}{4} - \frac{1}{2q}$, $b = \frac{3}{4} - \frac{1}{2q}$, $l = \frac{7}{4} - \frac{1}{2q}$.

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